Flat steady states in stellar dynamics—existence and stability

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Abstract

We consider a special case of the three dimensional Vlasov-Poisson system where the particles are restricted to a plane, a situation that is used in astrophysics to model extremely flattened galaxies. We prove the existence of steady states of this system. They are obtained as minimizers of an energy-Casimir functional from which fact a certain dynamical stability property is deduced. From a mathematics point of view these steady states provide examples of partially singular solutions of the three dimensional Vlasov-Poisson system.

1 Introduction

In astrophysics the time evolution of large stellar systems such as galaxies is often modeled by the Vlasov-Poisson system:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0,$$

$$\triangle U = 4\pi \rho,\ \lim_{x \to \infty} U(t,x) = 0,$$

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v) dv.$$

Here $f = f(t,x,v) \geq 0$ denotes the density of the stars in phase space, $t \in \mathbb{R}$ denotes time, $x, v \in \mathbb{R}^3$ denote position and velocity respectively, ρ is the spatial mass density, and U the gravitational potential. The only interaction between the stars is via the gravitational field which the stars create collectively, in particular, collisions are neglected. When modeling an extremely flattened galaxy the stars can be taken to be concentrated in a plane (the (x_1, x_2)-plane). The corresponding potential which is given by the usual integral representation induces a force field which accelerates the particles only parallelly to the plane, and the Vlasov-Poisson system takes the form

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0,\ t \in \mathbb{R},\ x, v \in \mathbb{R}^2, \tag{1.1}$$

$$U(t,x) = -\int_{\mathbb{R}^2} \frac{\rho(t,y)}{|x-y|} dy, \tag{1.2}$$

$$\rho(t,x) = \int_{\mathbb{R}^2} f(t,x,v) dv. \tag{1.3}$$

Note that from this point on, $x, v \in \mathbb{R}^2$. The three dimensional phase space and spatial densities are given as

$$\tilde{f}(t,x,x_3,v,v_3) = f(t,x,v)\delta(x_3)\delta(v_3)$$

and

$$\tilde{\rho}(t,x,x_3) = \rho(t,x)\delta(x_3)$$

where δ denotes the Dirac distribution. It should be emphasized that the system (1.1), (1.2), (1.3) is not a two dimensional version of the Vlasov-Poisson system but a special case of the three dimensional system with partially singular phase space density. In the present paper we are concerned with the existence of steady states of this system and with their stability properties. There are a number of aspects which make this problem interesting. Although such flat solutions of the Vlasov-Poisson system occur as models in the astrophysics literature, cf. [4, 6], we know of no mathematical investigation of this situation. The fact that the distribution function is singular in the x_3-direction, or, alternatively, that the two dimensional Vlasov equation is coupled to a potential with the three dimensional $1/|x|$-singularity, makes this problem mathematically nontrivial. We refer to [16], where solutions of the Vlasov-Poisson system which are measures are treated in the one dimensional case; an extension of these results to higher dimensions is not known.

Finally, the method that we employ to study the existence and the stability properties of steady states was recently used in a spherically symmetric, regular, three dimensional situation in [11]. The present paper demonstrates that this method extends beyond the case of spherical symmetry, although this assumption played an important role in [11].

To see how steady states of the system (1.1), (1.2), (1.3) can be obtained, note first that if $U = U(x)$ is time independent, the particle energy

$$E = \frac{1}{2}|v|^2 + U(x) \tag{1.4}$$

is conserved along characteristics of (1.1). Thus any function of the form

$$f(x,v) = \phi(E) \tag{1.5}$$

satisfies the Vlasov equation. We construct steady states as minimizers of an appropriately defined energy-Casimir functional. Given a function $Q = Q(f) \geq 0,\ f \geq 0$, we define

$$\mathcal{D}(f) := \int\!\!\int Q(f)\,dv\,dx + \frac{1}{2}\int\!\!\int |v|^2 f\,dv\,dx + \frac{1}{2}\int \rho_f U_f\,dx.$$

Here $f = f(x,v)$ is taken from some appropriate set $\mathcal{F}_M$ of functions which in particular have total mass equal to a prescribed constant M, ρ_f denotes the spatial density induced by f via (1.3), and U_f denotes the potential induced by ρ_f via (1.2). If one can show that the functional $\mathcal{D}$ has a minimizer, then the corresponding Euler-Lagrange equation turns out to be of the form (1.5). An alternative way to obtain steady states would be to substitute (1.5) into (1.3) so that ρ would become a functional of U, and it would remain to solve (1.2) which becomes a nonlinear integral equation for U. This route is followed for example in [1] for the regular three dimensional problem. The major difficulty then is to show that the resulting steady state has finite mass and compact support—properties which are essential for a steady state to qualify as a physically viable model—, and this problem has been dealt with for the polytropic ansatz

$$f(x,v) = (E - E_0)_+^\mu$$

where E_0 is a constant, $-1 < \mu < 7/2$, and $(\cdot)_+$ denotes the positive part. Our approach has the advantage that finiteness of the total mass and compact

support are built in or appear naturally, and these properties do not depend on a specific ansatz like the polytropic one. Furthermore, the fact that the steady state is obtained as a minimizer of the functional $\mathcal{D}$ implies a certain nonlinear stability property of that steady state.

The paper proceeds as follows. In the next section the assumptions on the function Q which determines our energy-Casimir functional are stated, and some preliminary results, in particular a lower bound of $\mathcal{D}$ on $\mathcal{F}_M$, are established. The main difficulties in finding a minimizer of $\mathcal{D}$ arise from the fact that $\mathcal{D}$ is neither positive definite nor convex, and from the lack of compactness: Along a minimizing sequence some mass might escape to infinity. However, using the scaling properties of $\mathcal{D}$ and a certain splitting estimate we show that along a minimizing sequence the total mass has to concentrate in a disc of a certain radius R_M, depending only on M. In [11] the corresponding argument required the assumption of spherical symmetry. In the present paper we only require axial symmetry with respect to the x_3-axis. The corresponding estimates are proved in Section 3 and are used in Section 4 to show the existence of a minimizer. It is then straight forward to show that the Euler-Lagrange equation is (equivalent to) (1.5), thereby completing the existence proof for the steady states. The resulting stability property of such steady states is discussed in Section 5. Since we have to restrict our functions in the set $\mathcal{F}_M$ to axially symmetric ones stability holds only with respect to such perturbations, and also perturbations transversal to the (x_1, x_2)-plane are not covered. Moreover, the stability result is only conditional in the sense that so far no existence theory for the initial value problem for the flat Vlasov-Poisson system is available. To obtain a complete stability result, global existence of solutions which preserve the energy-Casimir functional $\mathcal{D}$ would be needed, at least for data close to the steady states. In the last section we briefly discuss the regularity properties of the obtained steady states.

We conclude this introduction with some references to the literature. In the regular three dimensional situation the existence of global classical solutions to the corresponding initial value problem has been shown in [17], cf. also [14, 15, 21]. The existence of steady states for the case of the polytropic ansatz was investigated in [1] and [3]. We refer to [6] for contributions to the stability problem in the astrophysics literature. As to mathematically rigorous results on the stability problem, we mention [10, 11] for applications of the present approach in the regular, three dimensional case, cf. also [22].

An investigation of linearized stability is given in [2]. For the plasma physics case, where the sign in the Poisson equation is reversed, the stability problem is much easier and better understood. We refer to [5, 12, 13, 19]. A plasma physics situation with magnetic field is investigated in [9].

2 Preliminaries; a lower bound for $\mathcal{D}$

We first state the assumptions on Q which we need in the following:
Assumptions on Q: For $Q \in C^1([0,\infty[)$, $Q \geq 0$, and constants $C_1,\dots,C_4 > 0$, $F_0 > 0$, and $0 < \mu_1, \mu_2, \mu_3 < 1$ consider the following assumptions:

(Q1) $Q(f) \geq C_1 f^{1+1/\mu_1},\ f \geq F_0$,

(Q2) $Q(f) \leq C_2 f^{1+1/\mu_2},\ 0 \leq f \leq F_0$,

(Q3) $Q(\lambda f) \geq \lambda^{1+1/\mu_3} Q(f),\ f \geq 0,\ 0 \leq \lambda \leq 1$,

(Q4) $Q''(f) > 0,\ f > 0$, and $Q'(0) = 0$.

(Q5) $C_3 Q''(f) \leq Q''(\lambda f) \leq C_4 Q''(f)$ for $f > 0$ and λ in some neighborhood of 1.

The above assumptions imply that Q' is strictly increasing with range $[0,\infty[$, and we denote its inverse by q, i. e.,

$$Q'(q(\epsilon)) = \epsilon,\ \epsilon \geq 0; \tag{2.1}$$

we extend q by $q(\epsilon) = 0,\ \epsilon < 0$.
Remark: The steady states obtained later will be of the form

$$f_0(x,v) = q(E_0 - E)$$

with some $E_0 < 0$ and E as defined in (1.4). If we take $Q(f) = f^{1+1/\mu},\ f \geq 0$, this leads to the polytropic ansatz, and such a Q satisfies the assumptions above if $0 < \mu < 1$. If we take

$$Q(f) = C_1 f^{1+1/\mu_1} + C_2 f^{1+1/\mu_2} \tag{2.2}$$

with $0 < \mu_1, \mu_2 < 1$ and constants $C_1, C_2 > 0$ then again the above assumptions hold, but q is not of polytropic form. Due to the assumption of axial symmetry which we will have to make for other reasons,

$$L_3 = x_1 v_2 - x_2 v_1,$$

the x_3-component of angular momentum, is conserved along characteristics as well. It would be a purely technical matter to allow for depence on L_3 of a type where for example the constants in (2.2) could be replaced by functions of L_3 which are bounded and bounded away from 0. We refer to [11] for the necessary modifications.

For a measurable function $f = f(x,v)$ we define

$$\rho_f(x) := \int f(x,v)\, dv$$

and

$$U_f := -\frac{1}{|\cdot|} * \rho_f;$$

as to the existence of this convolution see Lemma 2 below. Then define

$$\begin{aligned}
E_{\mathrm{kin}}(f) &:= \frac{1}{2} \int\int |v|^2 f(x,v)\, dv\, dx,\\
E_{\mathrm{pot}}(f) &:= \frac{1}{2} \int \rho_f(x) U_f(x)\, dx = -\frac{1}{2} \int\int \frac{\rho_f(x)\rho_f(y)}{|x-y|} dx\, dy,\\
\mathcal{C}(f) &:= \int\int Q(f(x,v))\, dv\, dx,\\
\mathcal{P}(f) &:= E_{\mathrm{kin}}(f) + \mathcal{C}(f),\\
\mathcal{D}(f) &:= \mathcal{P}(f) + E_{\mathrm{pot}}(f).
\end{aligned}$$

The sum $E_{\mathrm{kin}}(f) + E_{\mathrm{pot}}(f)$ is the total energy corresponding to f, a conserved quantity for the time dependent problem, and the same is true for $\mathcal{C}$, a Casimir functional which corresponds to the conservation of phase space volume. $\mathcal{D}$ is the energy-Casimir functional, and $\mathcal{P}$ is the positive part of that functional. We will also use the notation U_ρ and $E_{\mathrm{pot}}(\rho)$ if $\rho = \rho(x)$ is not necessarily induced by some $f = f(x,v)$. The following two sets will serve as domains of definition for the energy-Casimir functional $\mathcal{D}$:

$$\mathcal{F}_M := \left\{ f \in L^1(\mathrm{I\!R}^4) \,|\, f \geq 0,\ \int\int f\, dv\, dx = M,\ \mathcal{P}(f) < \infty \right\}, \tag{2.3}$$

where $M>0$ is prescribed, and

$$\mathcal{F}_M^S := \Big\{ f \in \mathcal{F}_M \,|\, f \text{ is axially symmetric} \Big\}. \tag{2.4}$$

Here axial symmetry means that

$$f(Ax, Av) = f(x,v),\; x,v \in \mathbb{R}^2,\; A \in \mathrm{SO}(2).$$

When viewed as a function on the effective phase space $\mathbb{R}^4$, f is spherically symmetric, but when viewed as a function over the full phase space $\mathbb{R}^6$, f is only axially symmetric. The induced potential does not share the properties of spherically symmetric potentials which is why we prefer the phrase axially symmetric. We do not restrict ourselves to the set $\mathcal{F}_M^S$ from the beginning in order to point out where exactly the symmetry is needed.

The aim of the present section is to establish a lower bound for $\mathcal{D}$ of a form that will imply the boundedness of $\mathcal{P}$ along any minimizing sequence.

Lemma 1 *Let (Q1) hold and let $n_1 = 1+\mu_1$. Then there exists a constant $C>0$ such that for all $f \in \mathcal{F}_M$,*

$$\int \rho_f^{1+1/n_1}\, dx \le C\,(M + \mathcal{P}(f)).$$

Proof : We split the v integral into small and large v's and optimize to obtain the estimate

$$\rho_f(x) \le C \left(\int f^{1+1/\mu_1} dv \right)^{2\mu_1/(4+2\mu)} \left(\int |v|^2 f\, dv \right)^{2/(4+2\mu_1)}.$$

By definition of n_1 and assumption (Q1) we find

$$\begin{aligned} \int \rho_f^{1+1/n_1} dx &\le C \left(\int f^{1+1/\mu_1} dv\, dx + \int |v|^2 f\, dv\, dx \right) \\ &\le C \left(F_0^{1+1/\mu_1} \int f dv\, dx + \frac{1}{C_1} \int Q(f)\, dv\, dx + \int |v|^2 f\, dv\, dx \right), \end{aligned}$$

and by definition of $\mathcal{P}$ this is the assertion. □

Note that $3/2 < 1 + 1/n_1 < 2$, and since by definition $\rho_f \in L^1(\mathbb{R}^2)$ for $f \in \mathcal{F}_M$, we have $\rho_f \in L^{4/3}(\mathbb{R}^2)$ for $f \in \mathcal{F}_M$.

Lemma 2 *If* $\rho \in L^{4/3}(\mathbb{R}^2)$ *then* $U_\rho \in L^4(\mathbb{R}^2)$*, and there exists a constant* $C > 0$ *such that for all* $\rho \in L^{4/3}(\mathbb{R}^2)$*,*

$$\|U_\rho\|_4 \leq C\|\rho\|_{4/3}, \; -E_{\rm pot}(\rho) \leq C\|\rho\|_{4/3}^2.$$

Proof : The assertion follows from generalized Young's inequality [18, p. 32], since $1/|\cdot| \in L^2_w(\mathbb{R}^2)$, the weak L^2-space, and from Hölder's inequality. $\square$

Combining the previous two lemmata yields the desired lower bound of $\mathcal{D}$ over the set $\mathcal{F}_M$:

Lemma 3 *Let* Q *satisfy assumption (Q1). Then*

$$\mathcal{D}_M := \inf\{\mathcal{D}(f) \,|\, f \in \mathcal{F}_M\} > -\infty,$$

and there exists a constant $C_M > 0$ *depending on* M *such that*

$$\mathcal{D}(f) \geq \mathcal{P}(f) - C_M\Big(1 + \mathcal{P}(f)^{n_1/2}\Big), \; f \in \mathcal{F}_M,$$

and for any minimizing sequence $(f_n) \subset \mathcal{F}_M$ *of* $\mathcal{D}$ *we have*

$$\mathcal{P}(f_n) \leq C_M, \; n \in \mathbb{N}.$$

Proof : If we interpolate the $L^{4/3}$-norm between the L^1-norm and the L^{1+1/n_1}-norm and apply Lemma 1 we find

$$\int \rho_f^{4/3} dx \leq C_M \left(\int \rho_f^{1+1/n_1} dx\right)^{n_1/3} \leq C_M (1 + \mathcal{P}(f))^{n_1/3}.$$

Thus by Lemma 2

$$\mathcal{D}(f) \geq \mathcal{P}(f) - C_M\Big(1 + \mathcal{P}(f)\Big)^{2\frac{n_1}{3}\frac{3}{4}} \geq \mathcal{P}(f) - C_M\Big(1 + \mathcal{P}(f)^{n_1/2}\Big).$$

Since $n_1 < 2$ the rest of the lemma is obvious after possibly choosing C_M larger. $\square$

In later sections we will have to assume axial symmetry, and we will need the fact that

$$\mathcal{D}_M^S := \inf\{\mathcal{D}(f) \,|\, f \in \mathcal{F}_M^S\} > -\infty,$$

which of course follows from the previous lemma.

3 Scaling and Splitting

The behaviour of $\mathcal{D}$ and M under scaling transformations can be used to relate the $\mathcal{D}_M$'s for different values of M:

Lemma 4 *Let Q satisfy the assumptions (Q1)–(Q3). Then $-\infty < \mathcal{D}_M < 0$ for each $M > 0$, and for all $0 < M_1 \leq M_2$,*

$$\mathcal{D}_{M_1} \geq \left(\frac{M_1}{M_2}\right)^{1+\alpha} \mathcal{D}_{M_2},$$

where $\alpha = 1/(1-\mu_3) > 0$. The same assertions hold for $\mathcal{D}_M^S$ instead of D_M.

Proof : Given any function $f(x,v)$, we define a rescaled function $\bar{f}(x,v) = af(bx,cv)$, where a, b, $c > 0$. Then

$$\int\!\!\int \bar{f}\, dv\, dx = ab^{-2}c^{-2} \int\!\!\int f\, dv\, dx \tag{3.1}$$

and

$$\mathcal{D}(\bar{f}) = b^{-2}c^{-2}\mathcal{C}(af) + ab^{-2}c^{-4}E_{\mathrm{kin}}(f) + a^2b^{-3}c^{-4}E_{\mathrm{pot}}(f). \tag{3.2}$$

Proof of $\mathcal{D}_M < 0$: Fix some $f \in \mathcal{F}_1^S$ with compact support and $f \leq F_0$, and let $a = Mb^2c^2$ so that $\bar{f} \in \mathcal{F}_M^S$. The last term in $\mathcal{D}(\bar{f})$ is negative and of the order b, and we want to make this term dominate the others as $b \to 0$. Choose $c = b^{-\gamma/2}$ so that $a = Mb^{2-\gamma}$, and assume that $a \leq 1$ so that $af \leq F_0$. By (Q2),

$$\mathcal{D}(\bar{f}) \leq C\left(b^{(2-\gamma)/\mu_2} + b^{\gamma}\right) - \overline{C}b$$

where C, $\overline{C} > 0$ depend on f. Since we want the last term to dominate as $b \to 0$, we need $\gamma > 1$ and $(2-\gamma)/\mu_2 > 1$, and, in order that $a \leq 1$ as $b \to 0$, also $\gamma < 2$. Such a choice of γ is possible since $\mu_2 < 1$, and thus $\mathcal{D}(\bar{f}) < 0$ for b sufficiently small.
Proof of the scaling inequality: Assume that $f \in \mathcal{F}_{M_2}$ and $\bar{f} \in \mathcal{F}_{M_1}$ so that by (3.1),

$$ab^{-2}c^{-2} = \frac{M_1}{M_2} =: m \leq 1. \tag{3.3}$$

By (3.2) and (Q3),

$$\mathcal{D}(\bar{f}) \geq ma^{1/\mu_3}\mathcal{C}(f) + mc^{-2}E_{\mathrm{kin}}(f) + m^2bE_{\mathrm{pot}}(f)$$

provided $a \leq 1$. Now we require that

$$ma^{1/\mu_3} = mc^{-2} = m^2 b.$$

Together with (3.3) this determines a, b, c in terms of m. In particular,

$$a = m^{\mu_3/(1-\mu_3)} \leq 1$$

as required and

$$\mathcal{D}(\bar{f}) \geq m^{1+\frac{1}{1-\mu_3}} \mathcal{D}(f).$$

Since for any given choice of a, b, c the mapping $f \mapsto \bar{f}$ is one-to-one and onto between $\mathcal{F}_{M_2}$ and $\mathcal{F}_{M_1}$ as well as between $\mathcal{F}^S_{M_2}$ and $\mathcal{F}^S_{M_1}$ the scaling inequality follows. $\Box$

The following two lemmata are crucial in proving that along a minimizing sequence the mass concentrates in a certain ball. It is here that we need the additional symmetry assumption and where the estimates become more involved than in the regular spherically symmetric case. The aim is to estimate the effect on $\mathcal{D}$ of splitting the matter distribution into a part inside a ball B_R of (large) radius R about 0 and a part outside.

Lemma 5 *There exists a constant $C > 0$ such that for every $\rho \in L^1 \cap L^{4/3}(\mathbb{R}^2)$ which is nonnegative and axially symmetric, i. e., $\rho(x) = \rho(|x|)$, and every $R > 0$ the following estimate holds:*

$$-\int_{|x|>R} \rho(x)\, U_\rho(x)\, dx \leq C R^{-1/2} \|\rho\|_{4/3} \int_{|x|>R} \rho(x)\, dx.$$

Proof : Due to the symmetry of ρ the potential is given by

$$U_\rho(x) = U_\rho(r) = -4 \int_0^\infty \frac{s}{r+s} \rho(s)\, K\left(\frac{2\sqrt{rs}}{r+s}\right) ds$$

where the elliptic integral K is defined as

$$K(\xi) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\xi^2 \sin^2\phi}} = \int_0^1 \frac{dt}{\sqrt{1-\xi^2 t^2}\sqrt{1-t^2}},\ 0 \leq \xi < 1.$$

We need to estimate the singularity in K:

$$
\begin{aligned}
K(\xi) &\leq \int_0^1 \frac{dt}{\sqrt{1-\xi t}\sqrt{1-t}} = \frac{1}{\sqrt{\xi}} \ln \frac{1+\sqrt{\xi}}{1-\sqrt{\xi}} \\
&\leq C\,(1-\ln(1-\xi)),\ 0 \leq \xi < 1.
\end{aligned}
$$

Substituting for ξ yields

$$
1-\xi = \frac{(\sqrt{r}-\sqrt{s})^2}{r+s} = \frac{s}{r+s}\left(1-\sqrt{r/s}\right)^2 \geq \frac{1}{2}\left(1-\sqrt{r/s}\right)^2 \geq \frac{1}{8}(1-r/s)^2
$$

for $0 \leq r \leq s$; the case $r \geq s$ is analogous. Thus

$$
K\left(\frac{2\sqrt{rs}}{r+s}\right) \leq C\,(1-\ln(1-[r,s])),\ r,s>0 \tag{3.4}
$$

where

$$
[r,s] := \min\left\{\frac{r}{s}, \frac{s}{r}\right\}.
$$

Now

$$
-\int_{|x|>R} \rho(x) U_\rho(x)\,dx = 8\pi \int_R^\infty \int_0^\infty \rho(r)\rho(s)\frac{rs}{r+s} K\left(\frac{2\sqrt{rs}}{r+s}\right) dr\,ds = I_1 + I_2
$$

where in I_1 the variable r ranges in $[0,2s]$ and in I_2 it ranges in $[2s,\infty[$. Using (3.4) and Hölder's inequality we find

$$
\begin{aligned}
I_1 &\leq C \int_R^\infty \rho(s) \int_0^{2s} r\rho(r)\,(1-\ln(1-[r,s]))\,dr\,ds \\
&\leq C\|\rho\|_{4/3} \int_R^\infty \rho(s) \left(\int_0^{2s} r\Big(1-\ln(1-[r,s])\Big)^4 dr\right)^{1/4} ds \\
&\leq C\|\rho\|_{4/3} \int_R^\infty s^{1/2}\rho(s)\,ds \leq C\|\rho\|_{4/3} R^{-1/2} \int_R^\infty s\rho(s)\,ds;
\end{aligned}
$$

note that with $\sigma = r/s$,

$$
\begin{aligned}
\int_0^{2s} r\Big(1-\ln(1-[r,s])\Big)^4 dr &= s^2 \int_0^1 \sigma\Big(1-\ln(1-\sigma)\Big)^4 d\sigma \\
&\quad + s^2 \int_1^2 \sigma\Big(1-\ln(1-1/\sigma)\Big)^4 d\sigma \\
&= Cs^2,\ s>0.
\end{aligned}
$$

The second term is much easier to estimate: For $r > 2s$ we have $-\ln(1 - s/r) \leq \ln 2$, and by Hölder's inequality,

$$
\begin{aligned}
I_2 &\leq C \int_R^\infty s\rho(s) \int_{2s}^\infty \frac{r}{r+s} \rho(r)\, dr\, ds \leq C \int_R^\infty s\rho(s)\, ds \int_R^\infty \rho(r)\, dr \\
&\leq C \|\rho\|_{4/3} R^{-1/2} \int_R^\infty s\rho(s)\, ds.
\end{aligned}
$$

Together with the estimate for I_1 this completes the proof. $\Box$

Lemma 6 *Let Q satisfy the assumptions (Q1)–(Q3) and let $f \in \mathcal{F}_M^S$. Then*

$$
\mathcal{D}(f) - \mathcal{D}_M^S \geq \left(\frac{C_\alpha \mathcal{D}_M^S}{M^2} \int_{|x|<R} \int f\, dv\, dx - \frac{C_M}{\sqrt{R}} \right) \int_{|x|>R} \int f\, dv\, dx,\ R > 0,
$$

where the constant $C_\alpha < 0$ depends on α from Lemma 4 and $C_M > 0$ depends on M.

Proof : Let B_R denote the ball of radius R about 0 in $\mathbb{R}^2$, let $1_{B_R \times \mathbb{R}^2}$ be the characteristic function of $B_R \times \mathbb{R}^2$,

$$
f_1 = 1_{B_R \times \mathbb{R}^2} f,\ f_2 = f - f_1,
$$

and let ρ_i and U_i denote the induced spatial densities and potentials respectively, $i = 1, 2$. We abbreviate $\lambda = \int f_2$. Then

$$
\begin{aligned}
\mathcal{D}(f) &= \mathcal{P}(f_1) + \mathcal{P}(f_2) + \frac{1}{2} \int U_1 \rho_1\, dx + \frac{1}{2} \int U_2 \rho_2\, dx + \int U_1 \rho_2\, dx \\
&\geq \mathcal{D}_{M-\lambda}^S + \mathcal{D}_\lambda^S - C_M R^{-1/2} \lambda
\end{aligned}
$$

since $f_1 \in \mathcal{F}_{M-\lambda}^S$ and $f_2 \in \mathcal{F}_\lambda^S$. To estimate the "mixed term" in the potential energy we have used Lemma 5; note that for $f \in \mathcal{F}_M$, $\|\rho_f\|_{4/3}$ is bounded by a constant depending only on M, cf. Lemma 1. Since $\alpha > 0$, there is a constant $C_\alpha < 0$, such that

$$
(1-x)^{1+\alpha} + x^{1+\alpha} - 1 \leq C_\alpha (1-x) x,\ 0 \leq x \leq 1.
$$

Using Lemma 4 and noticing that $\mathcal{D}_M^S < 0$ we find that

$$
\begin{aligned}
\mathcal{D}(f) - \mathcal{D}_M^S &\geq \left[(1 - \lambda/M)^{1+\alpha} + (\lambda/M)^{1+\alpha} - 1 \right] \mathcal{D}_M^S - C_M R^{-1/2} \lambda \\
&\geq C_\alpha \mathcal{D}_M^S (1 - \lambda/M) \lambda/M - C_M R^{-1/2} \lambda
\end{aligned}
$$

which is the assertion. $\Box$

4 Minimizers of $\mathcal{D}$

Before we show the existence of a minimizer of $\mathcal{D}$ over the set $\mathcal{F}_M^S$ we use Lemma 6 to show that along a minimizing sequence the mass has to concentrate in a certain ball:

Lemma 7 *Let Q satisfy the assumptions (Q1)–(Q3), and define*

$$R_M := \left(\frac{2MC_M}{C_\alpha \mathcal{D}_M^S}\right)^2$$

where $C_\alpha < 0$ and $C_M > 0$ are as in Lemma 6. If $(f_n) \subset \mathcal{F}_M^S$ is a minimizing sequence of $\mathcal{D}$, then for any $R > R_M$,

$$\lim_{n\to\infty} \int_{|x|\geq R} \int f_n dv\, dx = 0.$$

Proof : If not, there exist some $R > R_M$, $\lambda > 0$, and a subsequence, called (f_n) again, such that

$$\lim_{n\to\infty} \int_{|x|\geq R} \int f_n dv\, dx = \lambda.$$

For every $n \in \mathbb{N}$ we can now choose $R_n > R$ such that

$$\lambda_n := \int_{|x|\geq R_n} \int f_n dv\, dx = \frac{1}{2} \int_{|x|\geq R} \int f_n dv\, dx.$$

Then

$$\lim_{n\to\infty} \int_{|x|\geq R_n} \int f_n dv\, dx = \lim_{n\to\infty} \lambda_n = \lambda/2 > 0.$$

Applying Lemma 6 to B_{R_n} we get

$$\begin{aligned}\mathcal{D}(f_n) - \mathcal{D}_M^S \geq & \left(\frac{C_\alpha \mathcal{D}_M^S}{M^2}(M-\lambda_n) - \frac{C_M}{\sqrt{R_n}}\right)\lambda_n > \left(\frac{C_\alpha \mathcal{D}_M^S}{M^2}(M-\lambda_n) - \frac{C_M}{\sqrt{R}}\right)\lambda_n \\ & \to \left(\frac{C_\alpha \mathcal{D}_M^S}{M^2}(M-\lambda/2) - \frac{C_M}{\sqrt{R}}\right)\frac{\lambda}{2} \geq \left(\frac{C_\alpha \mathcal{D}_M^S}{2M} - \frac{C_M}{\sqrt{R}}\right)\frac{\lambda}{2}\end{aligned}$$

as $n \to \infty$, since $0 < \lambda \leq M$. By definition of R_M the expression in the parenthesis is positive for $R > R_M$, and this contradicts the fact that (f_n) is a minimizing sequence. □

As a further prerequisite for the existence proof of a minimizer we establish a compactness property of the potential energy functional:

Lemma 8 *Let* $(\rho_n) \subset L^{3/2} \cap L^1(\mathbb{R}^2)$ *be bounded and axially symmetric with*

$$\rho_n \rightharpoonup \rho_0 \text{ weakly in } L^{3/2}(\mathbb{R}^2),\ n \to \infty.$$

Then

$$E_{\rm pot}(\rho_n - \rho_0) \to 0 \text{ and } E_{\rm pot}(\rho_n) \to E_{\rm pot}(\rho_0),\ n \to \infty.$$

Proof : We consider the convergence of $E_{\rm pot}(\rho_n - \rho_0)$ first. By Lemma 5 and Hölder's inequality

$$\begin{aligned} |E_{\rm pot}(\rho_n - \rho_0)| &\le \left| \int (U_{\rho_n} - U_{\rho_0})(\rho_n - \rho_0)\, dx \right| \\ &\le \|\rho_n - \rho_0\|_{4/3} \|U_{\rho_n,R} - U_{\rho_0,R}\|_{L^4(B_R)} + \frac{C}{\sqrt{R}}, \end{aligned}$$

for any $R > 0$, where

$$U_{\rho,R}(x) := -\int_{|y|\le R} \frac{\rho(y)}{|x-y|} dy,\ x \in \mathbb{R}^2.$$

Thus it suffices to show that for $R > 0$ fixed the mapping

$$T : L^{3/2}(B_R) \ni \rho \mapsto 1_{B_R}(\rho * k) \in L^4(\mathbb{R}^2)$$

is compact where $k := 1_{B_{2R}} 1/|\cdot|$; note that we may cut off the Green's function as indicated since only x, y with $|x|, |y| \le R$ need to be considered. We use the Frechét-Kolmogorov criterion to show that T is compact. Let $S \subset L^{3/2}(B_R)$ be bounded. Then TS is bounded in $L^4(\mathbb{R}^2)$ by Lemma 2. Since the elements in TS have a uniformly compact support it remains to show that for $\rho \in S$,

$$(T\rho)_h \to T\rho \text{ in } L^4(\mathbb{R}^2),\ h \to 0,$$

where $g_h := g(\cdot + h),\ h \in \mathbb{R}^2$. But by Young's inequality,

$$\|(T\rho)_h - T\rho\|_4 \le \|\rho * (k_h - k)\|_4 \le \|\rho\|_{3/2} \|k_h - k\|_{12/7} \to 0$$

uniformly on S as $h \to 0$, since $k \in L^{12/7}(\mathbb{R}^2)$. Since

$$E_{\rm pot}(\rho_n) - E_{\rm pot}(\rho_0) = -\int U_{\rho_0}(\rho_n - \rho_0) - E_{\rm pot}(\rho_n - \rho_0)$$

the fact that $U_{\rho_0} \in L^4(\mathbb{R}^2)$ together with the weak convergence of ρ_n implies the remaining assertion. $\square$

Theorem 1 *Let Q satisfy the assumptions (Q1)–(Q4), and let $(f_n) \subset \mathcal{F}_M^S$ be a minimizing sequence of $\mathcal{D}$. Then there is a minimizer $f_0 \in \mathcal{F}_M^S$ and a subsequence (f_{n_k}) such that $\mathcal{D}(f_0) = \mathcal{D}_M^S$, $\operatorname{supp} f_0 \subset B_{R_M} \times \mathrm{IR}^2$ with R_M as in Lemma 7, and $f_{n_k} \rightharpoonup f_0$ weakly in $L^{1+1/\mu_1}(\mathrm{IR}^4)$. Furthermore, $E_{\mathrm{pot}}(f_{n_k} - f_0) \to 0$.*

Proof : By Lemma 3, $(\mathcal{P}(f_n))$ is bounded. Let $p_1 = 1 + 1/\mu_1$. Then the sequence (f_n) is bounded in $L^{p_1}(\mathrm{IR}^4)$ by assumption (Q1). Thus there exists a weakly convergent subsequence, denoted by (f_n) again, i. e.,

$$f_n \rightharpoonup f_0 \text{ weakly in } L^{p_1}(\mathrm{IR}^4).$$

Clearly, $f_0 \geq 0$ a. e., and f_0 is axially symmetric. Since by Lemma 7

$$\begin{aligned} M &= \lim_{n\to\infty} \int_{|x|\leq R_1} \int_{|v|\leq R_2} f_n dv\, dx + \lim_{n\to\infty} \int_{|x|\leq R_1} \int_{|v|\geq R_2} f_n dv\, dx \\ &\leq \lim_{n\to\infty} \int_{|x|\leq R_1} \int_{|v|\leq R_2} f_n dv\, dx + \frac{C}{R_2^2} \end{aligned}$$

where $R_1 > R_M$ and $R_2 > 0$ are arbitrary, it follows that

$$\int_{|x|\leq R_1} \int f_0 dv\, dx = M$$

for every $R_1 > R_M$. This proves the assertion on $\operatorname{supp} f_0$ and $\iint f_0 = M$. Also by weak convergence

$$\iint |v|^2 f_0 dv\, dx \leq \liminf_{n\to\infty} \iint |v|^2 f_n dv\, dx < \infty. \tag{4.1}$$

By Lemma 1 $(\rho_n) = (\rho_{f_n})$ is bounded in $L^{1+1/n_1}(\mathrm{IR}^2)$ where $n_1 = \mu_1 + 1$. After extracting a further subsequence, we thus have that

$$\rho_n \rightharpoonup \rho_0 := \rho_{f_0} \text{ weakly in } L^{3/2}(\mathrm{IR}^2),$$

and Lemma 8 implies the convergence of the potential energy term.

It remains to show that f_0 is actually a minimizer, in particular, $\mathcal{P}(f_0) < \infty$ so that $f_0 \in \mathcal{F}_M^S$. By Mazur's Lemma there exists a sequence $(g_n) \subset L^{p_1}(\mathrm{IR}^4)$ such that $g_n \to f_0$ strongly in $L^{p_1}(\mathrm{IR}^4)$ and g_n is a convex

combination of $\{f_k \,|\, k \geq n\}$. In particular, $g_n \to f_0$ a. e. on $\mathbb{R}^4$. By (Q4) the functional $\mathcal{C}$ is convex. Combining this with Fatou's Lemma implies that

$$\mathcal{C}(f_0) \leq \liminf_{n\to\infty} \mathcal{C}(g_n) \leq \limsup_{n\to\infty} \mathcal{C}(f_n).$$

Together with (4.1) this implies that

$$\mathcal{P}(f_0) \leq \lim_{n\to\infty} \mathcal{P}(f_n) < \infty;$$

note that $\lim_{n\to\infty} \mathcal{P}(f_n)$ exists. Therefore,

$$\mathcal{D}(f_0) = \mathcal{P}(f_0) + E_{\mathrm{pot}}(f_0) \leq \lim_{n\to\infty} (\mathcal{P}(f_n) + E_{\mathrm{pot}}(f_n)) = \mathcal{D}_M^S,$$

and the proof is complete. $\square$

Theorem 2 *Let Q satisfy the assumptions (Q1)–(Q5), and let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{D}$. Then*

$$f_0(x,v) = q(E_0 - E) \;\; a.\ e.\ on\ \mathbb{R}^4,$$

where

$$E = \frac{1}{2}|v|^2 + U_0(x),$$

$$E_0 = \frac{1}{M} \int\!\!\int (Q'(f_0) + E)\, f_0 \, dv\, dx < 0,$$

U_0 is the potential induced by f_0, and q is as defined in (2.1).

Note that $U_0 = U_{f_0}$ by construction, and f_0 is a function of the particle energy only and thus a steady state of the system (1.1), (1.2), (1.3). The regularity of U_0 and thus the sense in which f_0 satisfies the Vlasov equation (1.1) is investigated in the last section.
Proof : Let f_0 be a minimizer. For fixed $\epsilon > 0$ let $\eta : \mathbb{R}^4 \to \mathbb{R}$ be measurable, with compact support, axially symmetric, and such that

$$|\eta| \leq 1, \text{ a. e. on } \mathbb{R}^4,\ \eta \geq 0 \text{ a. e. on } \mathbb{R}^4 \setminus \operatorname{supp} f_0,$$

and

$$\epsilon \leq f_0 \leq \frac{1}{\epsilon} \text{ a. e. on } \operatorname{supp} f_0 \cap \operatorname{supp} \eta.$$

Below we will occasionally argue pointwise on $\mathbb{R}^4$ so we choose a representative of f_0 satisfying the previous estimate pointwise on $\operatorname{supp} f_0 \cap \operatorname{supp} \eta$. For

$$0 \leq h \leq \frac{\epsilon}{2(1+\|\eta\|_1)}$$

we define

$$g(h) = M \frac{h\eta + f_0}{\|h\eta + f_0\|_1}.$$

This defines a variation of f_0 with $g(h) \in \mathcal{F}_M^S$ and $g(0) = f_0$; note that

$$M - \frac{\epsilon}{2} \leq \|h\eta + f_0\|_1 \leq M + \frac{\epsilon}{2}.$$

We expand $\mathcal{D}(g(h)) - \mathcal{D}(f_0)$ in powers of h:

$$\begin{aligned}\mathcal{D}(g(h)) - \mathcal{D}(f_0) = & \iint \Big(Q(g(h)) - Q(f_0)\Big)\, dv\, dx + \frac{1}{2}\iint |v|^2 (g(h) - f_0)\, dv\, dx \\ & + \iint U_0 (g(h) - f_0)\, dv\, dx + E_{\text{pot}}(g(h) - f_0). \end{aligned} \tag{4.2}$$

Since $g(h) \geq 0$ on $\mathbb{R}^4$, $g(h)$ is differentiable with respect to h, and we write $g'(h)$ for this derivative. Note that both $g(h)$ and $g'(h)$ are functions of $(x,v) \in \mathbb{R}^4$, but we suppress this dependence and obtain

$$\begin{aligned} g'(h) &= \frac{M}{\|h\eta + f_0\|_1}\eta - M\frac{h\eta + f_0}{\|h\eta + f_0\|_1^2}\iint \eta\, dv\, dx, \\ g''(h) &= -2\frac{M}{\|h\eta + f_0\|_1^2}\left(\iint \eta\, dv\, dx\right)\eta + 2M\frac{h\eta + f_0}{\|h\eta + f_0\|_1^3}\left(\iint \eta\, dv\, dx\right)^2. \end{aligned}$$

Now

$$g'(0) = \eta - \frac{1}{M}\left(\iint \eta\, dv\, dx\right) f_0 \tag{4.3}$$

and

$$|g''(h)| \leq C\,(|\eta| + f_0)$$

so that on $\mathbb{R}^4$,

$$|g(h) - f_0 - hg'(0)| \leq C h^2 (|\eta| + f_0);$$

in the following, constants denoted by C may depend on f_0, η, and ϵ but never on h. We can now estimate the last three terms in (4.2):

$$\iint |v|^2 (g(h)-f_0)\,dv\,dx = h\iint |v|^2 g'(0)\,dv\,dx + O(h^2), \tag{4.4}$$

$$\iint U_0 (g(h)-f_0)\,dv\,dx = h\iint U_0\, g'(0)\,dv\,dx + O(h^2), \tag{4.5}$$

$$|E_{\mathrm{pot}}(g(h)-f_0)| \le C\|\rho_{g(h)}-\rho_0\|_{4/3}^2 \le Ch^2. \tag{4.6}$$

For the last estimate we used Lemma 2 and the fact that

$$|\rho_{g(h)}(x)-\rho_0(x)| \le Ch\int (|\eta|+f_0)(x,v)\,dv.$$

It remains to estimate the first term in (4.2). Consider first a point $(x,v)\in \operatorname{supp} f_0$ with $f_0(x,v)>0$. Then

$$\begin{aligned} Q(g(h))-Q(f_0) &= Q'(f_0)(g(h)-f_0) + \frac{1}{2}Q''(\tau)(g(h)-f_0)^2 \\ &= hQ'(f_0)g'(0) + h^2\frac{1}{2}Q'(f_0)g''(\theta) + \frac{1}{2}Q''(\tau)(g(h)-f_0)^2 \end{aligned}$$

where τ lies between $g(h)$ and f_0 and θ lies between 0 and h; both τ and θ depend on (x,v). Thus

$$\Big|Q(g(h))-Q(f_0)-hQ'(f_0)g'(0)\Big| \le CQ'(f_0)\,(|\eta|+f_0)\,h^2 + CQ''(\tau)(|\eta|^2+f_0^2)\,h^2.$$

On $\operatorname{supp} f_0$ we have

$$\frac{1}{4}f_0 \le g(h) \le 2f_0$$

provided $0<\epsilon<\epsilon_0$ with $\epsilon_0>0$ sufficiently small. Thus τ lies between $f_0/4$ and $2f_0$, and by iterating (Q5) a finite, h-independent number of times we find

$$Q''(\tau) \le CQ''(f_0).$$

By (Q3) and (Q5),

$$\left(2^{1+1/\mu_3}-1\right)Q(f_0) \ge Q(2f_0)-Q(f_0) \ge Q'(f_0)\,f_0 + CQ''(f_0)\,f_0^2$$

and thus

$$\Big|Q(g(h))-Q(f_0)-hQ'(f_0)g'(0)\Big| \le CQ(f_0)h^2 + C|\eta|h^2;$$

here we used the continuity of Q' and Q'' and the fact that $\epsilon \leq f_0 \leq 1/\epsilon$ on $\mathrm{supp}\,\eta \cap \mathrm{supp}\, f_0$. The above estimate holds for any point $(x,v) \in \mathrm{supp}\, f_0$ with $f_0(x,v) > 0$. Now consider a point (x,v) with $f_0(x,v) = 0$. Then

$$g(h) = M \frac{h\eta}{\|h\eta + f_0\|_1} \leq C|\eta| h$$

so that by (Q4) and (Q2),

$$\begin{aligned} \left| Q(g(h)) - Q(f_0) - hQ'(f_0)g'(0) \right| = Q(g(h)) &\leq Q(Ch|\eta|) \\ &\leq C|\eta|^{1+1/\mu_2} h^{1+1/\mu_2} \end{aligned}$$

for $h > 0$ sufficiently small. Thus

$$\iint \left| Q(g(h)) - Q(f_0) - hQ'(f_0)g'(0) \right| dv\, dx \leq Ch^{1+\delta} \tag{4.7}$$

for some $\delta > 0$. Combining (4.4), (4.5), (4.6), and (4.7) with the fact that f_0 is a minimizer we find

$$0 \leq \mathcal{D}(g(h)) - \mathcal{D}(f_0) = h \iint \left(Q'(f_0) + \frac{1}{2}|v|^2 + U_0 \right) g'(0)\, dv\, dx + O(h^{1+\delta})$$

for all $h > 0$ sufficiently small. Recalling (4.3) and the definitions of E and E_0 this implies that

$$\iint \left(Q'(f_0) + E - E_0 \right) \eta\, dv\, dx \geq 0.$$

Recalling the class of admissable test functions η and the fact that $\epsilon > 0$ is arbitrary, provided it is sufficiently small, we conclude that

$$E - E_0 \geq 0 \quad \text{a. e. on } \mathbb{R}^4 \setminus \mathrm{supp}\, f_0$$

and

$$Q'(f_0) + E - E_0 = 0 \quad \text{a. e. on supp}\, f_0.$$

By definition of q—cf. (2.1)—this implies that

$$f_0(x,v) = q(E_0 - E) \text{ a. e. on } \mathbb{R}^4.$$

Since ρ_0 has compact support and $\lim_{x\to\infty} U_0(x) = 0$ we conclude that $E_0 < 0$. □

5 Dynamical Stability

We now discuss the dynamical stability of f_0. As noted in the introduction the existence of solutions to the initial value problem for the system (1.1), (1.2), (1.3) is open. In the following we therefore have to assume that for initial data in some (reasonably large) set $\mathcal{X} \subset \mathcal{F}_M^S$ the system has a solution $f(t)$ with $f(t) \in \mathcal{F}_M^S$ and $\mathcal{D}(f(t)) = \mathcal{D}(f(0))$, $t \geq 0$; classical solutions of the regular three dimensional Vlasov-Poisson system have these properties. The considerations below are only formal, and we emphasize this fact by not stating any theorems but only giving the stability estimates. First we note that for $f \in \mathcal{F}_M$,

$$\mathcal{D}(f) - \mathcal{D}(f_0) = d(f, f_0) + E_{\text{pot}}(f - f_0) \tag{5.1}$$

where

$$d(f, f_0) = \int\!\!\int \Big[Q(f) - Q(f_0) + (E - E_0)(f - f_0)\Big]\, dv\, dx.$$

Next we observe that $d(f, f_0) \geq 0$, $f \in \mathcal{F}_M$. For $E - E_0 \geq 0$ we have $f_0 = 0$, and thus

$$Q(f) - Q(f_0) + (E - E_0)(f - f_0) \geq Q(f) \geq 0.$$

For $E - E_0 < 0$,

$$Q(f) - Q(f_0) + (E - E_0)(f - f_0) = \frac{1}{2} Q''(\tilde{f})(f - f_0)^2 \geq 0 \tag{5.2}$$

provided $f > 0$; here $\tilde{f}$ is between f and f_0. If $f = 0$, the left hand side is still nonnegative by continuity. Now let Q satisfy the assumptions (Q1)–(Q5) and assume that the minimizer f_0 is unique in $\mathcal{F}_M^S$. Then we obtain the following stability estimate:

For every $\epsilon > 0$ there is $\delta > 0$ such that for any solution $f(t)$ of the flat Vlasov-Poisson system with $f(0) \in \mathcal{X}$,

$$d(f(0), f_0) + |E_{\text{pot}}(f(0) - f_0)| < \delta$$

implies

$$d(f(t), f_0) + |E_{\text{pot}}(f(t) - f_0)| < \epsilon, \ t \geq 0.$$

Assume this assertion were false. Then there exist $\epsilon_0 > 0$, $t_n > 0$, and $f_n(0) \in \mathcal{X}$ such that

$$d(f_n(0), f_0) + |E_{\text{pot}}(f_n(0) - f_0)| = \frac{1}{n}$$

but

$$d(f_n(t_n), f_0) + |E_{\rm pot}(f_n(t_n) - f_0)| \geq \epsilon_0 > 0.$$

From (5.1), we have $\lim_{n\to\infty}\mathcal{D}(f_n(0)) = \mathcal{D}_M^S$. Since $\mathcal{D}(f)$ is invariant under the assumed Vlasov-Poisson flow,

$$\lim_{n\to\infty}\mathcal{D}(f_n(t_n)) = \lim_{n\to\infty}\mathcal{D}(f_n(0)) = \mathcal{D}_M^S.$$

Thus, $(f_n(t_n)) \subset \mathcal{F}_M^S$ is a minimizing sequence of $\mathcal{D}$, and by Theorem 1 , we deduce that—up to a subsequence—$E_{\rm pot}(f_n(t_n) - f_0) \to 0$. Again by (5.1), $d(f_n(t_n), f_0) \to 0$, a contradiction.

Provided the assumed global Vlasov-Poisson flow is such that in addition $\|f(t)\|_\infty = \|f(0)\|_\infty$, $t \geq 0$, and that Q is such that

$$C_1 := \inf\{Q''(f) \,|\, 0 < f \leq C_0\} > 0$$

for some constant $C_0 > \|f_0\|_\infty$, then for $f(0) \leq C_0$ one obtains the stability estimate

$$\iint_{\mathbb{R}^4\setminus \mathrm{supp}\, f_0} Q(f(t))\,dv\,dx + \frac{C_1}{2}\iint_{\mathrm{supp}\, f_0} |f(t) - f_0|^2 dv\,dx + |E_{\rm pot}(f(t) - f_0)| < \epsilon.$$

This follows by estimating Q'' in the expansion (5.2) from below.

If the minimizer f_0 of $\mathcal{D}$ is not unique (and not isolated) in $\mathcal{F}_M^S$, then a solution starting close to f_0—in the sense of the above measurement for the deviation—remains close to the set of all minimizers in $\mathcal{F}_M^S$. In the regular, three dimensional case uniqueness of the minimizer can be shown for the polytropic ansatz, cf. [11].

6 Regularity

So far the steady states obtained in Section 4 satisfy the Vlasov-Poisson system (1.1), (1.2), (1.3) in a rather weak sense, in particular, the potential need not be sufficiently regular for characteristics of the Vlasov equation to exist so that the precise meaning of f_0 being a function of an invariant of the particle trajectories is questionable. The present section will remedy this under some very mild additional assumptions:

Theorem 3 *Assume that Q satisfies conditions (Q1)–(Q5), and in addition*

$$Q'(f) \geq C_1 f^{1/\mu_1},\ f \geq F_0.$$

Let $f_0 \in \mathcal{F}_M^S$ be a minimizer of $\mathcal{D}$ as obtained in Theorem 1, and ρ_0, U_0 the induced spatial density and potential respectively. Then ρ_0, $U_0 \in C^1(\mathbb{R}^2)$, and the first derivatives of U_0 are Hölder continuous. If the function q defined in (2.1) is locally Hölder continuous, then $U_0 \in C^2(\mathbb{R}^2)$, and the second derivatives of U_0 are Hölder continuous.

Proof : As a first step we wish to show that U_0 and ρ_0 are bounded. Recall that

$$-U_0(r) = 4\int_0^\infty \frac{s}{r+s}\rho_0(s)\,K(\xi)\,ds = I_1 + I_2,\ r \geq 0,$$

where in I_1 the variable s ranges in $[0,2r]$ and in I_2 it ranges in $[2r,\infty[$. Using the estimate (3.4) and the fact that $\rho_0 \in L^{3/2}(\mathbb{R}^2)$ we find

$$\begin{aligned} I_1 &\leq \frac{C}{r}\int_0^{2r} s\rho_0(s)\Big(1-\ln(1-[r,s])\Big)\,ds \\ &\leq \frac{C}{r}\|\rho_0\|_{3/2}\left(\int_0^2 rs\Big(1-\ln(1-[r,s])\Big)^3 ds\right)^{1/3} \leq Cr^{-1/3}. \end{aligned}$$

For $s \geq 2r$ the elliptic integral $K(\xi)$ is bounded, and again by Hölder's inequality we immediately obtain the same estimate for I_2 so that

$$|U_0(r)| \leq Cr^{-1/3},\ r > 0.$$

Next we know that

$$\rho_0(r) = \int q\left(E_0 - \frac{1}{2}v^2 - U_0(r)\right)dv = \begin{cases} 2\pi\displaystyle\int_{U(r)}^{E_0} q(E_0 - E)\,dE & , U_0(r) < E_0, \\ \qquad 0 & , U_0(r) \geq E_0. \end{cases}$$

The additional assumption on Q' implies that there are constants $C > 0$, $\epsilon_0 > 0$ such that

$$q(\epsilon) \leq C\epsilon^{\mu_1},\ \epsilon \geq \epsilon_0,$$

and this implies that

$$\rho_0(r) \leq Cr^{-(\mu_1+1)/3},\ r > 0.$$

Since we know that ρ_0 has compact support, it follows that $\rho_0 \in L^3(\mathbb{R}^2)$. We may now repeat the estimate for U_0 and obtain $I_1 \le Cr^{1/3}$ and $I_2 \le C$ so that U_0 and thus also ρ_0 are bounded.

For the rest of our argument we rely on the regularity properties of potentials generated by single layers. Firstly, the boundedness of ρ_0 implies that U_0 is Hölder continuous, cf. [7, page 42]. The relation between ρ_0 and U_0 immediately implies that ρ_0 shares this property. This implies that U_0 has Hölder continuous first order derivatives, a fact known as Ljapunov's Theorem, cf. [7, pages 66, 67]. Since

$$\rho_0'(r) = -2\pi q(E_0 - U_0(r))U_0'(r)$$

ρ_0 is continuously differentiable.

If q is locally Hölder continuous, then ρ_0 will have Hölder continuous first order derivatives; note that $E_0 - U_0(r)$ ranges in a bounded interval for $r \in [0,\infty[$ so the local Hölder continuity of q suffices. We can now apply Ljapunov's Theorem again and obtain the remaining assertions. $\Box$

We remark that above we considered U_0 as a function on the (x_1,x_2) plane. Of course the definition (1.2) makes perfect sense on all of $\mathbb{R}^3$, and as long as we consider only derivatives parallel to the (x_1,x_2) plane all the regularity assertions for U_0 hold on the whole space $\mathbb{R}^3$. However, it is well known that the derivative of U_0 perpendicular to the plane has a jump discontinuity on the plane.

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